Statistics 110 – Assignment 3 Solutions **Summer, 2006**

1. Rice 2.64

$$\begin{aligned} F_E(e) &= P[E \le e] = P[-\sqrt{2e/m} \le V \le \sqrt{2e/m}] \\ &= \Phi\left(\frac{\sqrt{2e/m}}{\sigma}\right) - \Phi\left(\frac{-\sqrt{2e/m}}{\sigma}\right) \end{aligned}$$

$$\begin{aligned} f_E(e) &= \frac{d}{de}\Phi\left(\frac{\sqrt{2e/m}}{\sigma}\right) - \Phi\left(\frac{-\sqrt{2e/m}}{\sigma}\right) \\ &= \phi\left(\frac{\sqrt{2e/m}}{\sigma}\right)\frac{1}{\sigma\sqrt{2em}} + \phi\left(\frac{-\sqrt{2e/m}}{\sigma}\right)\frac{1}{\sigma\sqrt{2em}} \\ &= 2\phi\left(\frac{\sqrt{2e/m}}{\sigma}\right)\frac{1}{\sigma\sqrt{2em}} \qquad \text{(by symmetry of } N(0,\sigma^2) \text{ density)} \\ &= \frac{2}{\sigma\sqrt{2\pi}}\frac{1}{\sigma\sqrt{2em}}\exp\left(\frac{-2e}{2m\sigma^2}\right) \\ &= \frac{1}{\sigma\sqrt{em\pi}}\exp\left(\frac{-e}{m\sigma^2}\right) \end{aligned}$$

2. Rice 2.66

   As shown in problem 7 of Assignment 2,

$$F(x) = 1 - \frac{1}{x^\alpha}$$

   So the quantile function $F^{-1}(p)$ satisfies,

$$\begin{aligned} p &= 1 - \frac{1}{x_p^\alpha} \\ &\Rightarrow 1 - p = x_p^\alpha \\ &\Rightarrow F^{-1}(p) = x_p = (1-p)^{1/\alpha} \end{aligned}$$

   Thus to generate a realization from this distribution, generate $U \sim Unif(0,1)$ and calculate

$$X = (1-U)^{1/\alpha}$$

   Note that if $U \sim Unif(0,1)$ so is $1-U$. So a realization can also be generated by

$$X = U^{1/\alpha}$$

3. Rice 2.67

   (a) The Weibull density function is

$$f(w) = F'(w) = e^{-(w/\alpha)^\beta} \frac{\beta w^{\beta-1}}{\alpha^\beta}, \;\; w \geq 0, \alpha \geq 0, \beta \geq 0,$$

   (b) For $x = g(w) = (w/\alpha)^\beta$, $w = g^{-1}(x) = \alpha x^{1/\beta}$

$$\frac{dw}{dx} = \frac{\alpha^\beta}{\beta w^{\beta-1}}$$

   then

$$f(x) = e^{-(x^{1/\beta})^\beta} \frac{\beta(\alpha x^{1/\beta})^{\beta-1}}{\alpha^\beta} \left| \frac{dw}{dx} \right| = e^{-x}$$

   This is the Exponential density function with parameter $\lambda = 1$. So $X \sim Exp(1)$.

   (c) As discussed earlier, if $U \sim Unif(0,1)$, then $X = -\log U \sim Exp(1)$. Then based on (b), $W = \alpha X^{1/\beta} \sim Weibell(\alpha, \beta)$.

4. Rice 2.68

   Radius: $R \sim Exp(1)$ and Area: $S = \pi R^2$. Then $R = \left(\frac{S}{\pi}\right)^{\frac{1}{2}}$ and $\frac{dR}{dS} = \frac{1}{2\sqrt{\pi S}}$. By using the formula of $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$, we have

$$f_S(s) = \lambda e^{-\lambda \sqrt{s/\pi}} \frac{1}{2\sqrt{\pi s}}.$$

   This can also be derived by

$$\begin{aligned} F_S(s) &= P[S \leq s] = P\left[X \leq \sqrt{\frac{s}{\pi}}\right] = F_R\left(\sqrt{\frac{s}{\pi}}\right) \\ &= 1 - \exp(-\lambda\sqrt{s/\pi}) \end{aligned}$$

   and

$$\begin{aligned} f_S(s) &= \frac{d}{ds} F_S(s) \\ &= \frac{d}{ds}(1 - \exp(-\lambda\sqrt{s/\pi})) \\ &= \lambda \exp(-\lambda\sqrt{s/\pi}) \frac{1}{2\sqrt{\pi s}} \end{aligned}$$

5. Rice 3.8

(a) i.

$$
\begin{aligned}
P[X > Y] &= \int_0^1 \int_y^1 \frac{6}{7}(x+y)^2 dxdy \\
&= \int_0^1 \frac{2}{7} \left[(x+y)^3 \Big|_y^1 \right] dy \\
&= \int_0^1 \frac{2}{7}[(1+y)^3 - 8y^3]dy \\
&= \frac{2}{7 \times 4} \; (1+y)^4 \Big|_0^1 - \frac{16}{7}\frac{1}{4} \; y^4 \Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}
$$

ii.

$$
\begin{aligned}
P[X + Y \le 1] &= \int_0^1 \int_0^{1-y} \frac{6}{7}(x+y)^2 dxdy \\
&= \int_0^1 \frac{2}{7} \left[(x+y)^3 \Big|_0^{1-y} \right] dy \\
&= \int_0^1 \frac{2}{7}(1 - y^3)dy \\
&= \frac{2}{7} \left(y - \frac{y^4}{4} \right) \Bigg|_0^1 \\
&= \frac{2}{7} - \frac{2}{7} \cdot \frac{1}{4} = \frac{3}{14}
\end{aligned}
$$

iii.

$$
\begin{aligned}
P[X \le \frac{1}{2}] &= \int_0^1 \int_0^{\frac{1}{2}} \frac{6}{7}(x+y)^2 dxdy \\
&= \int_0^1 \frac{2}{7} \left[(x+y)^3 \Big|_0^{\frac{1}{2}} \right] dy \\
&= \int_0^1 \frac{2}{7} \left[\left(y + \frac{1}{2} \right)^3 - y^3 \right] dy \\
&= \frac{1}{14} \left[\left(\frac{3}{2} \right)^4 - 1 - \left(\frac{1}{2} \right)^4 \right] = \frac{2}{7}
\end{aligned}
$$

(b)

$$f_X(x) = \int_0^1 \frac{6}{7}(x+y)^2 dy = \frac{2}{7}\left.(x+y)^3\right|_0^1$$
$$= \frac{2(x+1)^3 - 2x^3}{7}$$
$$= \frac{6x^2+6x+2}{7};\ 0 \le x \le 1.$$

$$f_Y(y) = \int_0^1 \frac{6}{7}(x+y)^2 dx = \frac{2}{7}\left.(x+y)^3\right|_0^1$$
$$= \frac{2(y+1)^3 - 2y^3}{7}$$
$$= \frac{6y^2+6y+2}{7};\ 0 \le x \le 1.$$

(c)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{6}{7}(x+y)^2}{\frac{2(y+1)^3-2y^3}{7}}$$
$$= \frac{3(x+y)^2}{(y+1)^3 - y^3}$$
$$= \frac{3(x+y)^2}{3y^2+3y+1};\ 0 \le x \le 1 \text{ and } 0 \le y \le 1.$$

Similarly (through symmetry)

$$f_{Y|X}(y|x) = \frac{3(x+y)^2}{3x^2+3x+1};\ 0 \le x \le 1 \text{ and } 0 \le y \le 1.$$

6. Rice 3.14

(a)

$$f_X(x) = \int_0^\infty xe^{-x(y+1)}dy$$
$$= e^{-x}\int_0^\infty xe^{-xy}dy \qquad \text{(Note that the integrand is the density of a Exp(x) RV)}$$
$$= e^{-x}$$

$$
\begin{aligned}
f_Y(y) &= \int_0^\infty xe^{-x(y+1)}dx \\
&= \frac{\Gamma(2)}{(y+1)^2}\int_0^\infty \frac{(y+1)^2}{\Gamma(2)}xe^{-x(y+1)}dx \\
&\qquad \text{(Note that the integrand is the density of a Gamma(2,y+1) RV)} \\
&= \frac{1}{(y+1)^2} \qquad (\Gamma(2)=1)
\end{aligned}
$$

X and Y are not independent since $f(x,y) \neq f_X(x)f_Y(y)$

(b)

$$
f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{xe^{-x(y+1)}}{1/(y+1)^2} = x(y+1)^2e^{-x(y+1)}
$$

This is the density of a $Gamma(2, y+1)$ RV

$$
f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy}
$$

This is the density of a $Exp(x)$ RV

7. Rice 3.20

$$
\begin{aligned}
f_{X_1,X_2}(x_1,x_2) &= f_{X_1}(x_1)f_{X_2|X_1}(x_2|x_1) = 1\frac{1}{x_1} \\
&= \frac{1}{x_1};\ 0 \le x_2 \le x_1 \le 1.
\end{aligned}
$$

$f_{X_1}(x_1) = 1$ and $X_1 \sim Unif(0,1)$.

$$
f_{X_2}(x_2) = \int_{x_2}^1 \frac{1}{x_1}dx_1 = \log\frac{1}{x_2};\ 0 \le x_2 \le 1.
$$

8. Rice 3.24

$f_P(p) = I_{0\le p\le 1}(p)$ and $f_{X|P}(x|p) = p^x(1-p)^{1-x}$. Then

$$
f(x,p) = I_{0\le p\le 1}(p)p^x(1-p)^{1-x}
$$

$$
\begin{aligned}
f_X(x) &= \int_0^1 p^x(1-p)^{1-x}dp \\
&= \begin{cases} \int_0^1 pdp = 0.5 & \text{for } x=1 \\ \int_0^1 1-pdp = 0.5 & \text{for } x=0 \end{cases}
\end{aligned}
$$

Then

$$f_{P|X}(p|x) = \frac{f(x,p)}{f_X(x)}$$
$$= \begin{cases} \frac{p}{0.5} = 2p & \text{for } x = 1 \\ \frac{1-p}{0.5} = 2(1-p) & \text{for } x = 0 \end{cases}$$

This can also be written in the form

$$f_{P|X}(p|x) = 2p^x(1-p)^{1-x}$$

9. Rice 3.40

X / Y	0	1	2
0	0	1	2
1	1	2	3
2	2	3	4

The array above are enumerations of all the possible sums corresponding to different pairs of X and Y. Each case is of equal chance. Then $P[X+Y=0] = \frac{1}{9}$, $P[X+Y=1] = \frac{2}{9}$, and $P[X+Y=2] = \frac{3}{9} = \frac{1}{3}$, $P[X+Y=3] = \frac{2}{9}$, $P[X+Y=4] = \frac{1}{9}$.

10. (a)

$$\int_0^\infty \int_0^{3y} c(3y-x)e^{-y}dxdy = \int_0^\infty c(9y^2e^{-y} - 4.5y^2e^{-y})dy$$
$$= c\int_0^\infty 4.5y^2e^{-y}dy = 9c = 1$$

So $c = \frac{1}{9}$.

(b)

$$f_X(x) = \int_{\frac{x}{3}}^\infty \frac{(3y-x)e^{-y}}{9}dy$$
$$= \frac{1}{3}\int_{\frac{x}{3}}^\infty ye^{-y}dy - \frac{x}{9}\int_{\frac{x}{3}}^\infty e^{-y}dy$$
$$= \left[\frac{1}{3}[-ye^{-y}|_{\frac{x}{3}}^\infty + \int_{\frac{x}{3}}^\infty e^{-y}dy\right] - \frac{x}{9}e^{-\frac{x}{3}}$$
$$= \frac{1}{3}e^{-\frac{x}{3}}, \text{ for } x \geq 0,$$

So $X \sim Exp(\frac{1}{3})$.

$$f_Y(y) = \int_0^{4y} \frac{(4y-x)e^{-y}}{16} dx = -\frac{(4y-x)^2 e^{-y}}{16}\Big|_0^{4y} = \frac{y^2 e^{-y}}{2}, \text{ for } y \geq 0.$$

So Y follows Gamma(3,1) distribution. Since $f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$, X and Y are not independent.

(c)

$$E[Y] = \int_0^{\infty} \frac{y^3 e^{-y}}{2} dy = 3$$

(d)

$$\begin{aligned} f_{Y|X=x}(y) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{\frac{1}{16}(4y-x)e^{-y}}{\frac{1}{4}e^{-\frac{x}{4}}} \\ &= \frac{1}{4}(4y-x)e^{-y+\frac{x}{4}}, \text{ for } y \geq \frac{x}{4} \end{aligned}$$

(e)

$$\begin{aligned} E[Y|X=x] &= \int_{\frac{x}{4}}^{\infty} y \cdot \frac{1}{4}(4y-x)e^{-y+\frac{x}{4}} y dy \\ &= \int_{\frac{x}{4}}^{\infty} y^2 e^{-y} e^{\frac{x}{4}} dy - \int_{\frac{x}{4}}^{+\infty} \frac{xy}{4} e^{-y+\frac{x}{4}} dy \\ &= -y^2 e^{-y} e^{\frac{x}{4}} |_{\frac{x}{4}}^{\infty} + \int_{\frac{x}{4}}^{\infty} 2 \cdot y e^{-y} e^{\frac{x}{4}} dy - \int_{\frac{x}{4}}^{\infty} \frac{x}{4} \cdot y e^{-y} e^{\frac{x}{4}} dy \\ &= \frac{x^2}{16} + (2 - \frac{x}{4}) e^{\frac{x}{4}} \int_{\frac{x}{4}}^{\infty} y e^{-y} dy \\ &= \frac{x^2}{16} - (2 - \frac{x}{4}) e^{\frac{x}{4}} [y e^{-y} |_{\frac{x}{4}}^{\infty} - \int_{\frac{x}{4}}^{\infty} e^{-y} dy] \\ &= 2 + \frac{x}{4} \end{aligned}$$

(f)

$$E[g(X)] = E\{E[Y|X=x]\} = \int_0^{+\infty} (2 + \frac{x}{4}) \cdot \frac{1}{4} e^{-\frac{x}{4}} dx = 3$$

(g)

$$E[g(X)] = E[E[Y|X=x]] = E[Y]$$

Suggested Problems

1. Rice 2.71

   Let X be a discrete random variable with $p_k = P[X = k]$. Then

$$F_X(k) = \sum_{i=-\infty}^{k} p_k = F_X(k-1) + p_k$$

   Thus

$$P[Y = k] = P[F_X(k-1) < U \leq F_X(k)] = F_X(k) - F_X(k-1) = p_k = P[X = k]$$

   For $Geom(p)$ RVs

$$P[X = k] = p(1-p)^{k-1}; k = 1, 2, \ldots \qquad F_k = P[X \leq k] = 1 - (1-p)^k; k = 0, 2, \ldots$$

   Then a $Geom(p)$ random variable can be simulated by generating $U \sim Unif(0, 1)$ and by setting $X = k$ where k satisfies

$$1 - (1-p)^{k-1} < U \leq 1 - (1-p)^k$$

2. Rice 3.15

   (a)

$$\begin{aligned}
1 &= \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} c\sqrt{1 - x^2 - y^2}dydx \qquad \text{(Let } x = r\cos\theta, y = r\sin\theta) \\
&= \int_0^1 \int_0^{2\pi} cr\sqrt{1 - r^2}d\theta dr \\
&= 2\pi \int_0^1 cr\sqrt{1 - r^2}dr \\
&= \frac{-2\pi c}{3}\left(1 - r^2\right)^{1.5}\Big|_0^1 = \frac{2\pi c}{3}
\end{aligned}$$

   So $c = \frac{3}{2\pi}$.

   (b) The plot of the joint density looks like a half sphere.

(c) Using the change of variables $x = r\cos\theta, y = r\sin\theta$,

$$
\begin{aligned}
P[X^2 + Y^2 \leq 0.5] = P[R^2 \leq 0.5] &= P\left[R \leq \frac{\sqrt{2}}{2}\right] \\
&= \int_0^{\sqrt{2}/2} \int_0^{2\pi} \frac{3}{2\pi} r\sqrt{1-r^2}\, d\theta dr \\
&= \int_0^{\sqrt{2}/2} 3r\sqrt{1-r^2}\, dr \\
&= -(1-r^2)^1.5\Big|_0^{\sqrt{2}/2} \\
&= 1 - 0.5^{1.5} = 0.6464 = \frac{2\sqrt{2}-1}{2\sqrt{2}}
\end{aligned}
$$

(d)

$$
\begin{aligned}
f_X(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\, dy \\
&= \frac{3}{2\pi}\left(\frac{y}{2}\sqrt{1-x^2-y^2} + \frac{(1-x^2)^2}{2}\arcsin\frac{y}{1-x^2}\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\right) \\
&= \frac{3}{2}(1-x^2);\ \ 0 \leq x \leq 1
\end{aligned}
$$

By symmetry,

$$
f_Y(y) = \frac{3}{2}(1-y^2);\ \ 0 \leq x \leq 1
$$

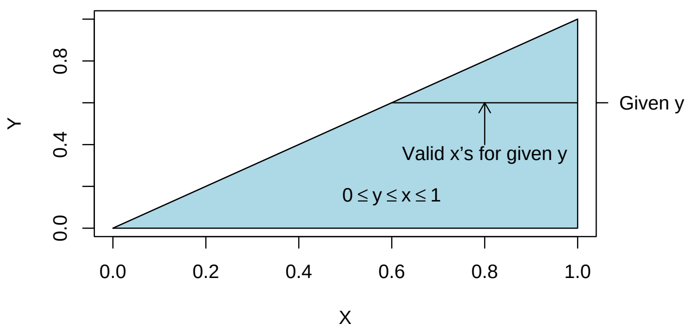
(e)

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{\frac{3}{2\pi}\sqrt{1-x^2-y^2}}{\frac{3}{2}(1-y^2)} \\
&= \frac{\sqrt{1-x^2-y^2}}{\pi(1-y^2)}; \qquad -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}
\end{aligned}
$$

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{\frac{3}{2\pi}\sqrt{1-x^2-y^2}}{\frac{3}{2}(1-x^2)} \\
&= \frac{\sqrt{1-x^2-y^2}}{\pi(1-x^2)}; \qquad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}
\end{aligned}
$$

3. Rice 3.18

(a) The region where the density is positive is



(b)

$$\begin{aligned}1 &= \int_0^1 \int_0^x k(x-y)dydx \\ &= \int_0^1 k\left(x^2 - \frac{x^2}{2}\right)dx \\ &= \int_0^1 k\frac{x^2}{2}dx \\ &= k\frac{1}{6}\end{aligned}$$

So $k = 6$

(c)

$$\begin{aligned}f_X(x) &= \int_0^x 6(x-y)dy \\ &= 3x^2; \qquad 0 \le x \le 1\end{aligned}$$

$$\begin{aligned}f_Y(y) &= \int_y^1 6(x-y)dx \\ &= 3x^2 - 6xy\Big|_y^1 \\ &= (3-6y) - (3y^2 - 6y^2) \\ &= 3 - 6y + 3y^2 \\ &= 3(1-y)^2; \qquad 0 \le y \le 1\end{aligned}$$

(d)

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{6(x-y)}{3x^2} \\
&= \frac{2(x-y)}{x^2}; \qquad 0 \le y \le x \le 1
\end{aligned}
$$

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{6(x-y)}{3(1-y)^2} \\
&= \frac{2(x-y)}{(1-y)^2}; \qquad 0 \le y \le x \le 1
\end{aligned}
$$

4. Rice 3.31

This algorithm stops when the finding the largest k such that $U < p_0 + p_1 + \ldots + p_k$. This is equivalent to finding the unique k satisfying $p_0 + \ldots + p_{k-1} \le U < p_0 + \ldots + p_k$, which has probability

$$
\begin{aligned}
P[X = k] &= P[p_0 + \ldots + p_{k-1} \le U < p_0 + \ldots + p_k] \\
&= p_k
\end{aligned}
$$