Distributions Related to the Normal

Statistics 110

Summer 2006

Distributions Related to the Normal

The are a number of very common distributions related to the normal distribution. These distributions underlie many common inference procedures in statistics.

- Chi-square distributions (χ^2_{df})

  Definition. *If* $Z \sim N(0,1)$*, then* $U = Z^2$ *has a* **Chi-square distribution with 1 degree of freedom** (χ^2_1)*.*

  The CDF of the of U is given by

$$F_U(u) = P[U \leq u] = P[-\sqrt{u} \leq Z \leq \sqrt{u}] = \Phi(\sqrt{u}) - \Phi(-\sqrt{u})$$

so the density is given by

$$\begin{aligned} f_U(u) &= \frac{d}{du}\left(\Phi(\sqrt{u}) - \Phi(-\sqrt{u})\right) \\ &= \frac{1}{2\sqrt{u}}\phi(\sqrt{u}) + \frac{1}{2\sqrt{u}}\phi(-\sqrt{u}) \\ &= \frac{1}{\sqrt{u}}\phi(\sqrt{u}) \\ &= \frac{u^{-1/2}}{2^{1/2}\sqrt{\pi}}e^{-u/2}; \qquad u \geq 0 \end{aligned}$$

Note that this happens to be the same as a $Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$ distribution.

Note that if $X \sim N(\mu, \sigma^2)$, then

$$(X - \mu)/\sigma \sim N(0, 1)$$

so

$$[(X - \mu)/\sigma]^2 \sim \chi_1^2$$

Definition. *If* $U_1, U_2, \ldots, U_n$ *are independent* χ^2_1 *RVs, then the distribution of* $V = U_1 + U_2 + \ldots + U_n$ *is a* **Chi-square distribution with n degrees of freedom** (χ^2_n).

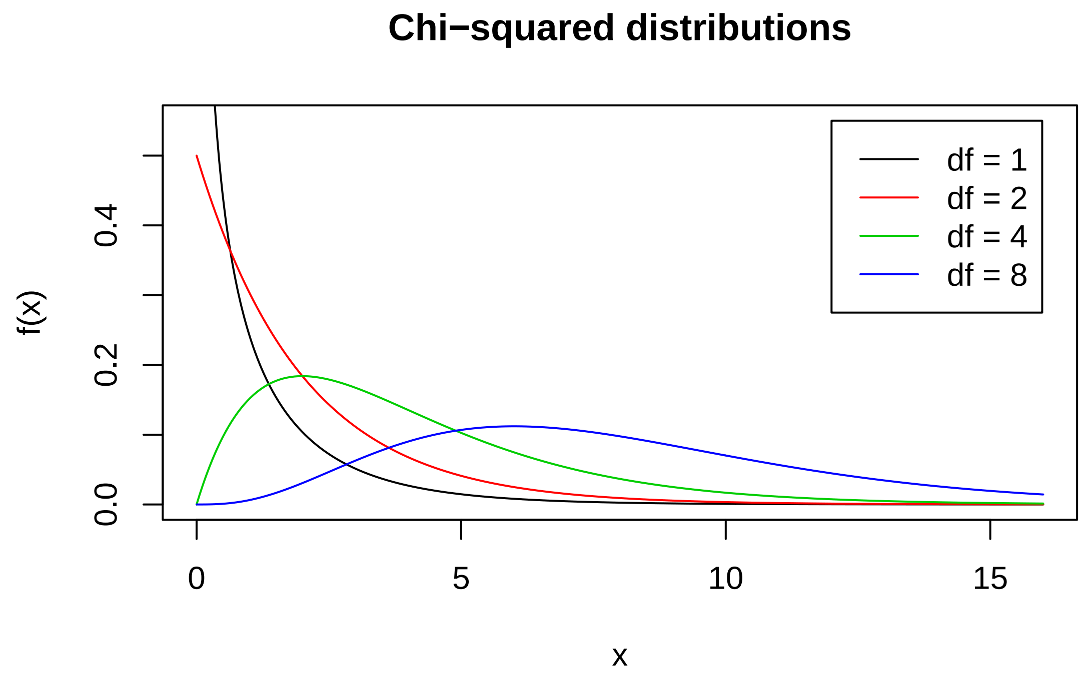
Note that this is the same as a $Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$, so the density is

$$f_V(v) = \frac{v^{(n/2)-1}}{2^{n/2}\Gamma(n/2)} e^{-v/2}; \qquad v \geq 0$$

and $E(V) = n, \mathrm{Var}(V) = 2n$. Also the MGF is

$$M_V(t) = \frac{1}{(1-2t)^{n/2}}$$

Since chi-squared random variables are special cases of gamma $(\lambda = \frac{1}{2})$, if $U \sim \chi^2_n$ is independent of $V \sim \chi^2_m$, then $U + V \sim \chi^2_{m+n}$.


Chi−squared distributions
df = 1
df = 2
df = 4
df = 8
f(x)
0.0
0.2
0.4
0
5
10
15
x

- t distributions (t_{df})

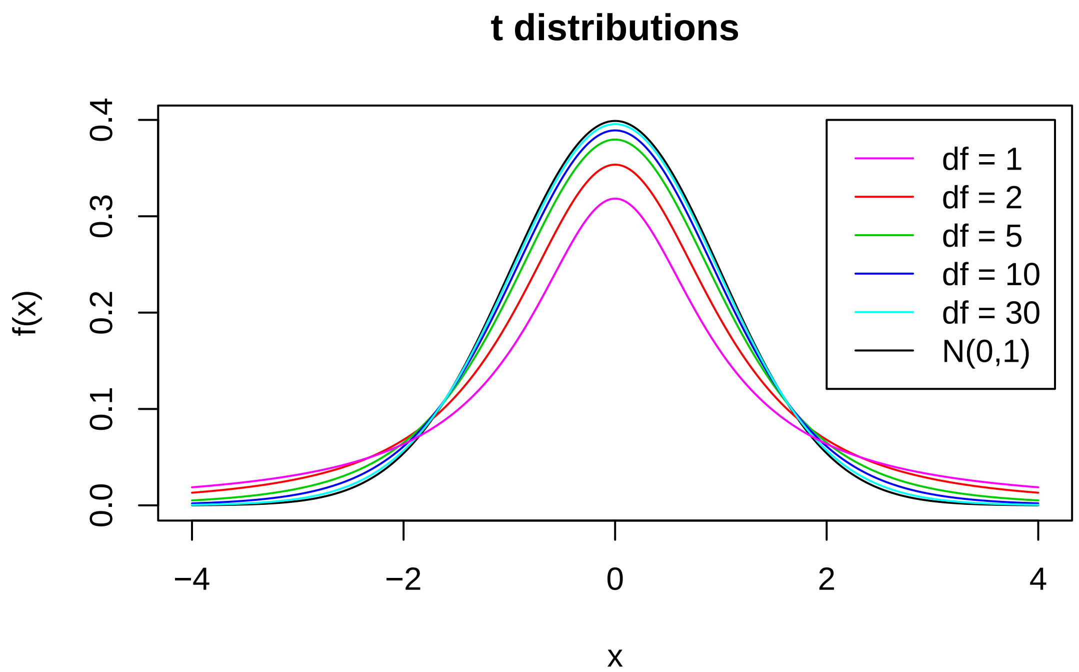
Definition. *If* $Z \sim N(0,1)$ *and* $U \sim \chi^2_n$*, and* Z *and* U *are independent, then the distribution of*

$$t = \frac{Z}{\sqrt{U/n}}$$

is called the $\boldsymbol{t}$ **distribution with** $\boldsymbol{n}$ **degrees of freedom** (t_n).

The density of the t_n distribution is

$$f_t(t) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\;\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$



Note that the t_1 distribution is the same as the Cauchy(0,1) distribution.

$E[t^r] < \infty$ only if $r < n$. So t_1 has no moments, t_2 has a mean but no variance. Every other t_n distribution with $n > 2$ has a mean and a variance with

$$E[t] = 0; \qquad \mathrm{Var}(t) = \frac{n}{n-2}$$

Since the t_n distributions don't have all moments, there is no MGF.

As can be easily seen, t_n distributions are symmetric about 0. Also their variance is greater than 1, but decreasing towards 1 as the degrees of freedom increases.

Also as $n \to \infty$, the t_n distribution converges to the $N(0, 1)$ distribution. In many cases, t-tables, such as the one in Rice, denote the standard normal distribution as t_∞.

- F distributions (F_{df_1,df_2})

Definition. *Let U and V be independent with chi-squared RVs with m and n degrees of freedom respectively. The distribution of*

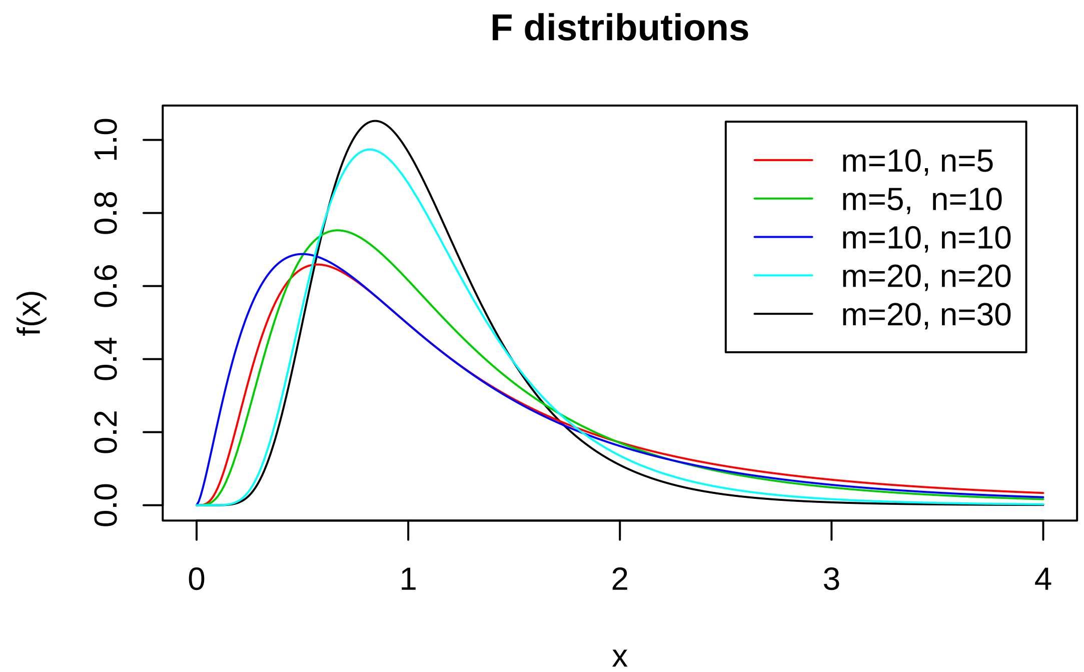
$$W = \frac{U/m}{V/n}$$

is called the $\boldsymbol{F}$ **distribution with** $\boldsymbol{m}$ **and** $\boldsymbol{n}$ **degrees of freedom** $(F_{m,n})$.

The density of the $F_{m,n}$ distribution is

$$f_F(w) = \frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} \left(\frac{m}{n}\right)^{m/2} w^{m/2-1} \left(1 + \frac{m}{n}w\right)^{-(m+n)/2}; \quad w \geq 0$$

$$E[W] = \frac{n}{n-2} \text{ for } n > 2; \quad \mathrm{Var}(W) = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)} \text{ for } n > 4$$



Note that if $W \sim F_{m,n}$, then $1/W \sim F_{n,m}$. Also the square of a t_n RV has an $F_{1,n}$ distribution.

Uses of $t_n, \chi^2_n, F_{m,n}$

Suppose that $X_1, X_2, \ldots, X_n$ are independent $N(\mu, \sigma^2)$ RVs.

- What is the distribution of the sample mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- What is the distribution of the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- What is their joint distribution?

These are important if we want to understand how large a sample is needed to estimate the population mean, μ, by the sample mean $\bar{X}$, to a given level of accuracy.

Theorem. *Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ RVs. Then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.*

Proof. As we have seen before, a linear combination of normals is normal. Therefore we only need to verify the moments.

$$E[\bar{X}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n} n\mu = \mu$$

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

□

Definition. $\mathbf{X} = \{X_1, \ldots X_n\}$ *has a multivariate normal* $N(\boldsymbol{\mu}, \Sigma)$ *where* $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_n]^T$ *is the vector of means* $(\mu_i = E[X_i])$ *and*

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2n} \\ \vdots & \vdots & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_n^2 \end{bmatrix}$$

is the Variance-Covariance matrix where $\sigma_i^2 = \mathrm{Var}(X_i)$ *and* $\sigma_{ij} = \mathrm{Cov}(X_i, X_j)$ *with density*

$$f(\mathbf{x}) = \frac{1}{|\Sigma|^{1/2}(2\pi)^{n/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)$$

where $\mathbf{x} = [x_1, \ldots, x_n]^T$*.*

Note the the form inside the exponential is proportional to

$$(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) \propto \sum_{i=1}^{n}\frac{(x_i-\mu_i)^2}{\sigma_i^2}+2\sum_{i<j}\frac{c_{ij}(x_i-\mu_i)(x_j-\mu_j)}{\sigma_i\sigma_j}$$

Lemma. *Let* $\{X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n\}$ *be multivariate normal* $N(\boldsymbol{\mu}, \Sigma)$ *RVs. Then* $\mathbf{X} = \{X_1, \ldots, X_m\}$ *is independent of* $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$ *if*

$$\mathrm{Cov}(X_i, Y_j) = 0 \text{ for all pairs } i, j$$

Proof. If $\mathrm{Cov}(X_i, Y_j) = 0$, it can be shown that $c_{ij} = 0$ (due to the block structure of Σ), so the terms

$$\frac{c_{ij}(x_i-\mu_i)(y_j-\mu_j)}{\sigma_i\sigma_j}$$

drop out of the sum. Rearranging the remaining terms allows the joint density to be factored as

$$f(\mathbf{x}, \mathbf{y}) = g(\mathbf{x})h(\mathbf{y})$$

where g is the density of $\mathbf{X}$ and h is the density of $\mathbf{Y}$. Since the density factors this way, $\mathbf{X}$ is independent of $\mathbf{Y}$. $\square$

Theorem. *Let* $X_1, X_2, \ldots, X_n$ *be iid* $N(\mu, \sigma^2)$ *RVs. Then* $\bar{X}$ *is independent of* $\{(X_1 - \bar{X}), \ldots, (X_n - \bar{X})\}$

Proof.

$$X_1 - \bar{X} = \left(1 - \frac{1}{n}\right) X_1 - \frac{1}{n} X_2 - \ldots - \frac{1}{n} X_n$$

The other terms $X_i - \bar{X}$ have an equivalent form.

$$
\begin{aligned}
\mathrm{Cov}(\bar{X}, X_1 - \bar{X}) &= \mathrm{Cov}\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \left(1 - \frac{1}{n}\right) X_1 - \frac{1}{n}\sum_{i=2}^{n} X_i\right) \\
&= \mathrm{Cov}\left(\frac{1}{n}X_1, \left(1 - \frac{1}{n}\right) X_1\right) + \sum_{i=2}^{n} \mathrm{Cov}\left(\frac{1}{n}X_i, \frac{-1}{n}X_i\right) \\
&= \frac{1}{n}\left(1 - \frac{1}{n}\right) \mathrm{Var}(X_1) - \sum_{i=2}^{n} \frac{1}{n^2}\mathrm{Var}(X_i) \\
&= \frac{1}{n}\left(\frac{n-1}{n}\right) \sigma^2 - (n-1)\frac{1}{n^2}\sigma^2 = 0
\end{aligned}
$$

Similarly $\mathrm{Cov}(\bar{X}, X_i - \bar{X}) = 0$ for the other i. Since these covariances are all 0, the independence result holds. $\square$

Corollary. $\bar{X}$ *and* S^2 *are independent.*

This holds since if X and Y are independent, so are $g(X)$ and $h(Y)$.

Theorem.

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Proof. First notice that

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_n$$

Now

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}[(X_i-\bar{X})+(\bar{X}-\mu)]^2$$

If we expand the square, using the fact that $\sum_{i=1}^{n}(X_i-\bar{X})=0$ we get

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2 + \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$$
$$= \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$$

Now notice that this is of the form $W = U + V$ and U and V are independent (since U is a function of S^2 and V is a function of $\bar{X}$).

So the MGFs satisfy $M_W(t) = M_U(t)M_V(t)$. Since $W \sim \chi^2_n$ and $V \sim \chi^2_1$, the MGF for U is

$$M_U(t) = \frac{M_W(t)}{M_V(t)} = \frac{(1-2t)^{-n/2}}{(1-2t)^{-1/2}} = \left(\frac{1}{1-2t}\right)^{(n-1)/2}$$

This is MGF for a χ^2_{n-1} $\square$

The term degrees of freedom come from the relationships such as $\sum_{i=1}^{n}(X_i - \bar{X}) = 0$. In this case, you can specify $n-1$ of the $\{X_i - \bar{X}\}$ to be whatever you want. However the remaining term is fixed $((X_n - \bar{X}) = -\sum_{i=1}^{n-1}(X_i - \bar{X}))$

Theorem.

$$E[\bar{X}] = \mu; \qquad E[S^2] = \sigma^2$$

($\bar{X}$ and S^2 are said to be unbiased estimates of μ and σ^2).

The first we've already proved and the second holds since

$$S^2 = \frac{\sigma^2 \chi^2_{n-1}}{n-1}$$

and $E[\chi^2_{n-1}] = n-1$.

Also these estimators get more precise as n increases as

$$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}; \qquad \mathrm{Var}(S^2) = \frac{2\sigma^4}{n-1}$$

If we want to perform inference on μ using $\bar{X}$ based on the results we have so far, we also need to know σ^2. It would be nice if we could do things without having to know this. In fact we can.

Theorem.

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Here we are standardizing by $S/\sqrt{n}$ instead of $\sigma/\sqrt{n}$.

Proof.

$$t = \frac{\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)}{\sqrt{S^2/\sigma^2}}$$

The numerator has a standard normal distribution and the denominator is the square root of a χ^2_{n-1} RV divided by its degree of freedom. $\square$

While we can make statements about $\bar{X}$without knowing σ^2, it does come at a cost.

As mentioned earlier, if $t \sim t_{n-1}, Z \sim N(0,1)$, $\mathrm{Var}(t) > \mathrm{Var}(Z)$.

In addition $P[-c \leq t \leq c] < P[-c \leq Z \leq c]$ for all $c > 0$. Thus you need bigger intervals to cover the same probability with ts than you do with normals.

We can expand some of these results to learn about difference between distributions. For example if $X_1, X_2, \ldots, X_m$ are iid $N(\mu_X, \sigma_X^2)$ and $Y_1, Y_2, \ldots, Y_n$ are iid $N(\mu_Y, \sigma_Y^2)$, then $\bar{X} - \bar{Y}$ is normally distributed with

$$E[\bar{X} - \bar{Y}] = \mu_X - \mu_Y; \qquad \mathrm{Var}(\bar{X} - \bar{Y}) = \frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}$$

Thus we can make probability statements about $\bar{X} - \bar{Y}$, an estimate of $\mu_X - \mu_Y$, assuming we know σ_X^2 and σ_Y^2.

One possible use of the final type of distribution mentioned, among many others, is to compare variances. Again, let $X_1, X_2, \ldots, X_m$ be iid $N(\mu_X, \sigma_X^2)$ and $Y_1, Y_2, \ldots, Y_n$ be iid $N(\mu_Y, \sigma_Y^2)$. If

$$S_X^2 = \frac{1}{m-1}\sum_{i=1}^{m}(X_i - \bar{X})^2; \qquad S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

then

$$\frac{S_X^2}{S_Y^2} \sim \frac{\sigma_X^2}{\sigma_Y^2} F_{m-1,n-1}$$

This distributional result allows use to make probability statements about $\frac{S_X^2}{S_Y^2}$, an estimate of variance ratio $\frac{\sigma_X^2}{\sigma_Y^2}$.

Making Probability Statements with $t_n, \chi^2_n, F_{m,n}$ Distributions

None of these distributions have nice CDF and quantile functions, so software or tables need to be to be used to make probability statements.

The tables that usually are presented are of the quantile function (as in done in the text). For example, for the t_n distribution, the quantiles which satisfy

$$P[T \leq t_p] = p$$

for different choices of p and n are available.

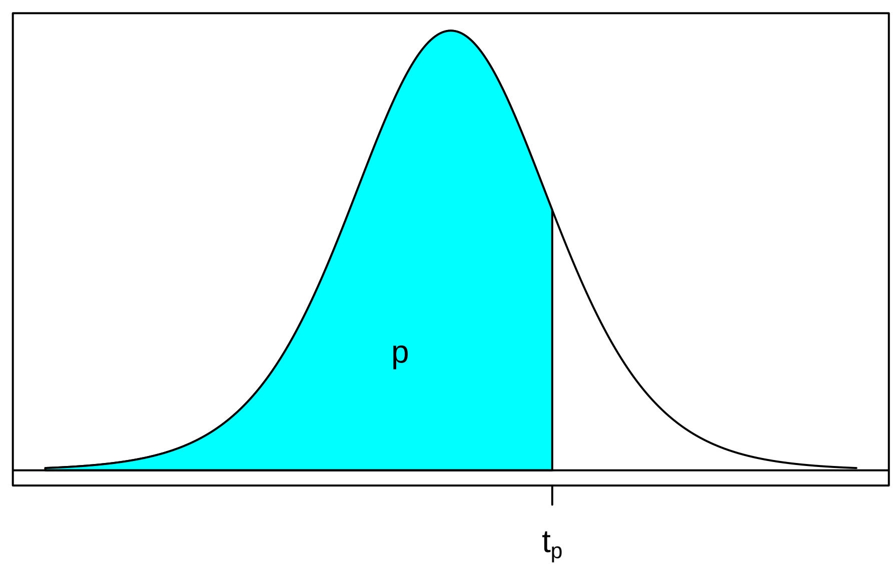


Table 4: Percentiles of t distribution

df	$t_{0.60}$	$t_{0.70}$	$t_{0.80}$	$t_{0.90}$	$t_{0.95}$	$t_{0.975}$	$t_{0.99}$	$t_{0.995}$
1	0.325	0.727	1.376	3.078	6.314	12.706	31.821	63.657
2	0.289	0.617	1.061	1.886	2.920	4.303	6.965	9.925
3	0.277	0.584	0.978	1.638	2.353	3.182	4.541	5.841
4	0.271	0.569	0.941	1.533	2.132	2.776	3.747	4.604
5	0.267	0.559	0.920	1.476	2.015	2.571	3.365	4.032
6	0.265	0.553	0.906	1.440	1.943	2.447	3.143	3.707
7	0.263	0.549	0.896	1.415	1.895	2.365	2.998	3.499
8	0.262	0.546	0.889	1.397	1.860	2.306	2.896	3.355
9	0.261	0.543	0.883	1.383	1.833	2.262	2.821	3.250
10	0.260	0.542	0.879	1.372	1.812	2.228	2.764	3.169
26	0.256	0.531	0.856	1.315	1.706	2.056	2.479	2.779
27	0.256	0.531	0.855	1.314	1.703	2.052	2.473	2.771
28	0.256	0.530	0.855	1.313	1.701	2.048	2.467	2.763
29	0.256	0.530	0.854	1.311	1.699	2.045	2.462	2.756
30	0.256	0.530	0.854	1.310	1.697	2.042	2.457	2.750
40	0.255	0.529	0.851	1.303	1.684	2.021	2.423	2.704
60	0.255	0.527	0.848	1.296	1.671	2.000	2.390	2.660
120	0.254	0.526	0.845	1.289	1.658	1.980	2.358	2.617
∞	0.253	0.524	0.842	1.282	1.645	1.960	2.326	2.576

For example we can determine the value c such that

$$P\left[\left|\frac{\bar{X}-\mu}{s/\sqrt{n}}\right| \leq c\right] = 0.95$$

i.e. how many standard errors will your sample average be from the true mean 95% of the time.

This is equivalent to

$$P\left[\frac{\bar{X}-\mu}{s/\sqrt{n}} \leq c\right] = 0.975$$

so $c = t_{0.975}$

For example if $n = 10, t_{0.975} = 2.228$

Note that as n increases, t_p decreases (and $p > 0.5$).

If you need to get quantiles for $p < 0.5$, use the fact that

$$t_{1-p} = -t_p$$

due to the symmetry of the t distribution.

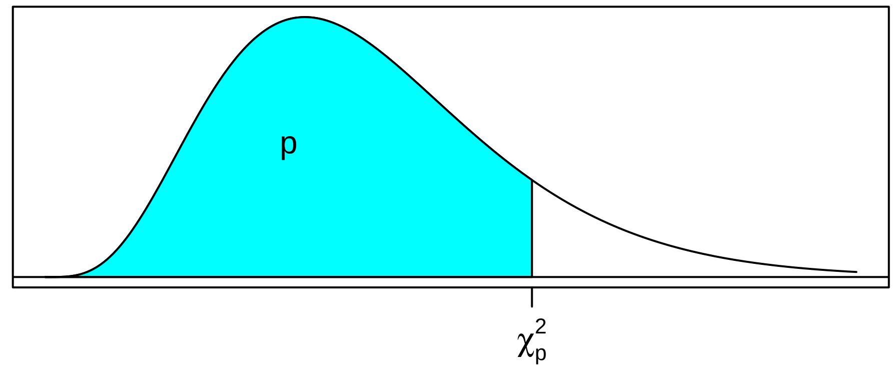
For example

$$t_{0.4} = -t_{0.6}$$

Also the last row of the table $n = \infty$ corresponds to the quantiles of the $N(0, 1)$ distribution, so instead of inverting the standard normal CDF table, the t-table can be used for some p.

For the χ^2_n distribution, the quantile functions are tabled for selected p and n.

$$P[X \le \chi^2_p] = p$$



When n increases, χ^2_p increases. This is not surprising, as this corresponds to adding more positive terms to get the distribution.

The table gives quantiles for small and large p since the χ^2_n distributions are skewed right.

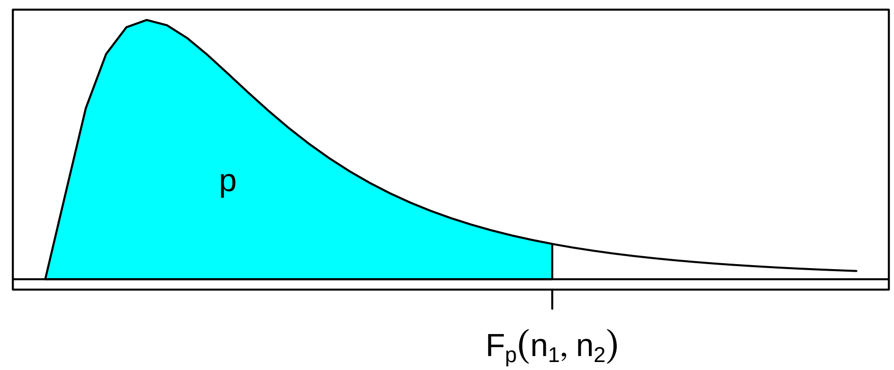
Table 3: Percentiles of χ^2 distribution

df	$\chi^2_{0.005}$	$\chi^2_{0.01}$	$\chi^2_{0.025}$	$\chi^2_{0.05}$	$\chi^2_{0.1}$	$\chi^2_{0.9}$	$\chi^2_{0.95}$	$\chi^2_{0.975}$	$\chi^2_{0.99}$	$\chi^2_{0.995}$
1	0.000039	0.00016	0.00098	0.0039	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.1026	0.2107	4.61	5.99	7.38	9.21	10.60
3	0.0717	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.34	12.84
4	0.207	0.297	0.484	0.711	1.06	7.78	9.49	11.14	13.28	14.86
5	0.412	0.554	0.831	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	0.676	0.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.95
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
24	9.88	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
120	83.85	86.92	91.57	95.70	100.62	140.23	146.57	152.21	158.95	163.65

The tables for the F_{n_1,n_2} are a bit different. While they still satisfy the general quantile relationship

$$P[F \leq F_p(n_1, n_2)] = p$$

there is a different table for for each p. The different entries correspond to different combinations of degrees of freedom.



When n_1 increases (with fixed n_2), $F_p(n_1, n_2)$ increases. However when n_2 increases (with fixed n_1), $F_p(n_1, n_2)$ decreases.

Table 5: Percentiles of F distribution - $F_{0.9}(n_1, n_2)$

n_2 \ n_1	1	2	3	4	5	6	7	8	9	10	40	60	120	∞
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	62.53	62.79	63.06	63.33
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.47	9.47	9.48	9.49
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.16	5.15	5.14	5.13
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.80	3.79	3.78	3.76
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.16	3.14	3.12	3.10
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.78	2.76	2.74	2.72
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.54	2.51	2.49	2.47
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.36	2.34	2.32	2.29
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.23	2.21	2.18	2.16
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.13	2.11	2.08	2.06
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.05	2.03	2.00	1.97
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	1.99	1.96	1.93	1.90
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	1.93	1.90	1.88	1.85
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	1.89	1.86	1.83	1.80
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	1.85	1.82	1.79	1.76
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.81	1.78	1.75	1.72
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.78	1.75	1.72	1.69
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.75	1.72	1.69	1.66
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.73	1.70	1.67	1.63
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.51	1.47	1.42	1.38
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.44	1.40	1.35	1.29
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.37	1.32	1.26	1.19
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.30	1.24	1.17	1.00

Table 5: Percentiles of F distribution - $F_{0.95}(n_1, n_2)$

n_2 \ n_1	1	2	3	4	5	6	7	8	9	10	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.03	1.98	1.93	1.88
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.39	1.32	1.22	1.00