

# STAT 221: STATISTICAL COMPUTING METHODS

Spring, 2004

Solution keys of ASSIGNMENT 3

Due on Apr. 9, 2004

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1. The given inequality will be rewritten as

$$\begin{aligned}\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} &\leq \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{1}{\alpha} \alpha x^{\alpha-1} + \frac{1}{\beta} \beta (1-x)^{\beta-1} \right) \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\alpha + \beta}{\alpha\beta} \left( \frac{\beta}{\alpha + \beta} g_1(x) + \frac{\alpha}{\alpha + \beta} g_2(x) \right)\end{aligned}$$

where  $g_1(x) = \alpha x^{\alpha-1}$  and  $g_2(x) = \beta(1-x)^{\beta-1}$ . That is, the proposing distribution is the mixture density and mass constant  $M = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\alpha+\beta}{\alpha\beta}$ . As explained in my section, the acceptance-rejection method is thus given by

- i. Generate  $x$  from  $g(x)$ .
  - (a) Generate  $u$  from  $\text{Unif}(0, 1)$ . If  $u \leq \beta/(\alpha + \beta)$ , let  $i = 1$ . Otherwise, let  $i = 2$ .
  - (b) Generate  $x$  from  $g_i(x)$  using the inverse cdf method.
- ii. Generate  $u$  from  $\text{Unif}(0, 1)$ . If  $u \leq f(x)/(Mg(x))$ , accept  $x$ . Otherwise, go back to step 1.

Your answer should explain full details of this algorithm.

2.

(a) The crude MC estimator is given by

$$\hat{\theta}_{\text{MC}} = \frac{1}{L} \sum_{\ell=1}^L \frac{e^{x^{(\ell)}} - 1}{e - 1}$$

with  $L$  draws of  $x$  from  $\text{Unif}(0, 1)$ .

(b) The importance sampling estimator is given by

$$\hat{\theta}_{\text{IS}} = \frac{1}{L} \sum_{\ell=1}^L \frac{e^{x^{(\ell)}} - 1}{e^{x^{(\ell)}}} = \frac{1}{L} \sum_{\ell=1}^L 1 - e^{-x^{(\ell)}}$$

with  $L$  draws of  $x$  from  $g(x) = e^x/(e - 1)$ . The inverse cdf method can be used to get draws from  $g(x)$ .

(c) We should find the optimal choice of control variates close to the integrand function. Using Taylor expansion  $e^x - 1 \approx x$ , the control variates estimator is given by

$$\hat{\theta}_{\text{CV}} = \frac{1}{L} \sum_{\ell=1}^L \frac{e^{x^{(\ell)}} - 1 - x^{(\ell)}}{e - 1} + \frac{1}{2(e - 1)}$$

with  $L$  draws of  $x$  from  $\text{Unif}(0, 1)$ .

(d) Because  $u$  and  $1 - u$  are antithetic each other when  $u \sim \text{Unif}(0, 1)$ , the antithetic variates estimator is given by

$$\hat{\theta}_{\text{AV}} = \frac{1}{L} \left( \sum_{\ell=1}^{L/2} \frac{e^{x^{(\ell)}} - 1}{e - 1} + \sum_{\ell=L/2+1}^L \frac{e^{1-x^{(\ell)}} - 1}{e - 1} \right)$$

with  $L$  draws of  $x$  from  $\text{Unif}(0, 1)$ .

Then the simulation results will be easily obtained.

**3.** The control variates estimator for the variance of the sample median can be obtained by

$$\begin{aligned} \text{Var}(X_{(n)}) &= \text{E}(X_{(n)})^2 = \text{E}(X_{(n)} - \bar{X} + \bar{X})^2 \\ &= \text{E}((X_{(n)} - \bar{X})^2 + 2\bar{X}(X_{(n)} - \bar{X}) + \text{E}(\bar{X})^2) \\ &\approx \frac{1}{L} \sum_{\ell=1}^L (X_{(n)}^{(\ell)} - \bar{X}^{(\ell)}) (X_{(n)}^{(\ell)} + \bar{X}^{(\ell)}) + \frac{1}{2n-1}. \end{aligned}$$

**4.** As shown in my section, the minimization problem of  $\text{Var}(\tilde{p}_b)$  hinges on minimizing

$$\begin{aligned} \text{E}_g \left( h(x) \frac{f(x)}{g(x)} \right)^2 &= \text{E}_g \left( I_{(z \geq c)} \frac{\phi(z)}{\phi(z-b)} \right)^2 \\ &= \int I_{(z \geq c)} \frac{\phi(z)^2}{\phi(z-b)} dz \\ &= \int I_{(z \geq c)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z+b)^2 + b^2} dz \\ &= e^{b^2} (1 - \Phi(z+b)). \end{aligned}$$

Solving the first order condition, we have

$$\frac{S(b+c)}{\phi(b+c)} = \frac{1}{2b}$$

and the Mills' ratio implies

$$\frac{1}{b+c} \left( 1 - \frac{1}{(b+c)^2} \right) \leq \frac{1}{2b} \leq \frac{1}{b+c}.$$

That is, the conditions for the optimal  $b^*$  are  $b \geq c$  and  $b^3 + cb^2 - (c^2 + 2)b - c^3 \leq 0$ . However, using the bisection method, we can find the optimal value of  $b$  for  $c = 2, 3$ , and  $4$ . Then the corresponding efficiency values can be easily computed.

$c$	$b^*$	efficiency
2	2.215930	19.0016
3	3.154849	221.9165
4	4.087179	6350.505