Sequential Importance Sampling (SIS)

AKA Particle Filtering, Sequential Imputation

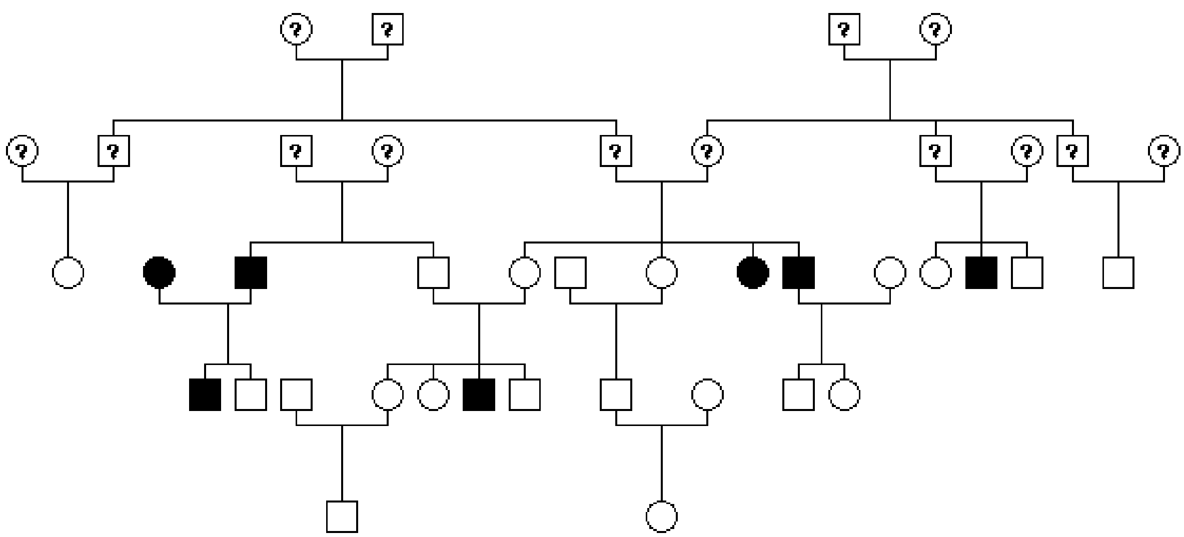
(Kong, Liu, Wong, 1994)

For many problems, sampling directly from the target distribution is difficult or impossible.

One reason possible reason for this is the size of the space that needs to be drawn from

Examples:

1) Linkage Analysis (Irwin, Cox, & Kong, 1994)



- m = 41 members
- n = 27 (nonfounders), f = 14 (founders)
- 8 markers from chromosome 19

- #alleles ranges from 6 to 8
- 14 members in top 2 generation have no marker data

Want to sample joint haplotypes for all pedigree members conditional on the observed marker and disease data

Assume that marker j has n_j possible alleles and the disease locus has two alleles.

Then the number of possible haplotypes for each person is

$$h = 4\prod n_j^2$$

and the maximum number of joint haplotypes possible is

$$H = h^m$$

If n_j = 8 for all markers, h = 1.1259×10^{15} and H = 1.29268×10^{617}.

Note that not all possible joint haplotypes included in H have positive probability since they won't be consistent with Mendelian segregration.

In addition the observed data will also reduce the number of possible haplotypes with positive probability.

2) Target tracking (Irwin, Cressie, & Johannesson, 2002)

Movement Model:

Position:

$$x_t = x_{t-1} + \bar{v}_{x,t}$$
$$y_t = y_{t-1} + \bar{v}_{y,t}$$

Velocity:

$$v_{x,t} = v_{x,t-1} + \delta_{x,t}$$
$$v_{y,t} = v_{y,t-1} + \delta_{y,t}$$

$$\bar{v}_{x,t} = \frac{v_{x,t} + v_{x,t-1}}{2} = v_{x,t-1} + \frac{1}{2}\delta_{x,t}$$
$$\bar{v}_{y,t} = \frac{v_{y,t} + v_{y,t-1}}{2} = v_{y,t-1} + \frac{1}{2}\delta_{y,t}$$

where $\delta_{x,t}$ and $\delta_{y,t}$ are the average accelerations in the x and y directions from time $t - 1$ to time t.

This gives

$$
\begin{aligned}
x_t &= x_{t-1} + \upsilon_{x,t-1} + \tfrac{1}{2}\delta_{x,t} \\
y_t &= y_{t-1} + \upsilon_{y,t-1} + \tfrac{1}{2}\delta_{x,t}
\end{aligned}
$$

This can be written in matrix format

$$X_t = GX_{t-1} + H\delta_t$$

where

$$X_t^T = \begin{bmatrix} x_t & y_t & \upsilon_{x,t} & \upsilon_{y,t} \end{bmatrix}$$

$$\delta_t^T = \begin{bmatrix} \delta_{x,t} & \delta_{x,t} \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \; H = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Assume that the model for the average accelerations is

$$\delta_t \sim N_2\left(0, \Lambda_t\right)$$

Measurement model:

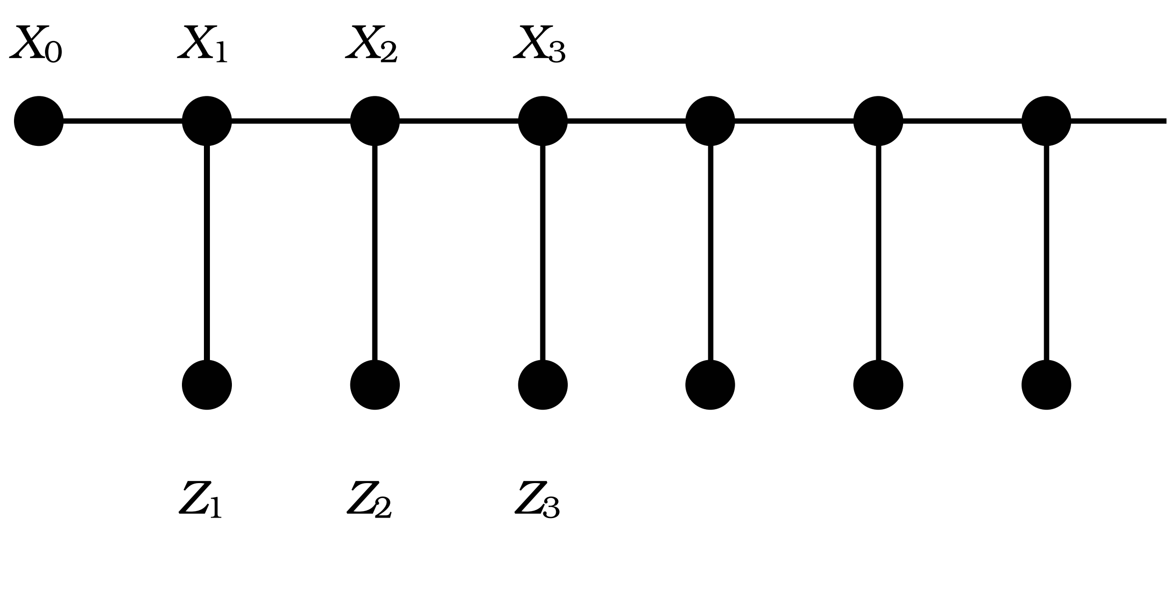
Two radars track the targets position with error

$$Z_t = FX_t + \varepsilon_t;\ \varepsilon_t \sim N\left(0, \Sigma_t\right)$$

where

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The probability structure can be described by the following graph



The model is an example of the hidden Markov model. The state variables X_t are described by a continuous state Markov Chain but are unobserved (hidden). All that is observed are the Z_t, the observed target positions

Problem:

Want to know the distribution of $X_t \middle| Z_{1:t}$ for each t ($Z_{1:t} = \left\{Z_1, \ldots, Z_t\right\}$).

Since this is a linear dynamic model, it can easily be solved by the Kalman filter (KF) (Kalman, 1960).

In this case, $X_t \middle| Z_{1:t}$ is Gaussian and the means and variances can be determined by the following simple update formulas.

$$\mu_t = E\left[X_t \middle| Z_{1:t}\right];\ \mu_{t|t-1} = E\left[X_t \middle| Z_{1:t-1}\right]$$

$$P_t = \mathrm{Var}\left(X_t \middle| Z_{1:t}\right);\ P_{t|t-1} = \mathrm{Var}\left(X_t \middle| Z_{1:t-1}\right)$$

Assuming $E\left[\delta_t\right] = E\left[\varepsilon_t\right] = 0$, the KF calculations are

$$
\begin{aligned}
\mu_{t|t-1} &= G\mu_{t-1} \\
P_{t|t-1} &= GP_tG^T + H\Lambda_t H^T \\
K_t &= P_{t|t-1}F^T\left[\Sigma_t + FP_{t|t-1}F^T\right]^{-1} \\
\mu_t &= \mu_{t|t-1} + K_t\left(Z_t - F\mu_{t|t-1}\right) \\
P_t &= P_{t|t-1} - K_tF_tP_{t|t-1}
\end{aligned}
$$

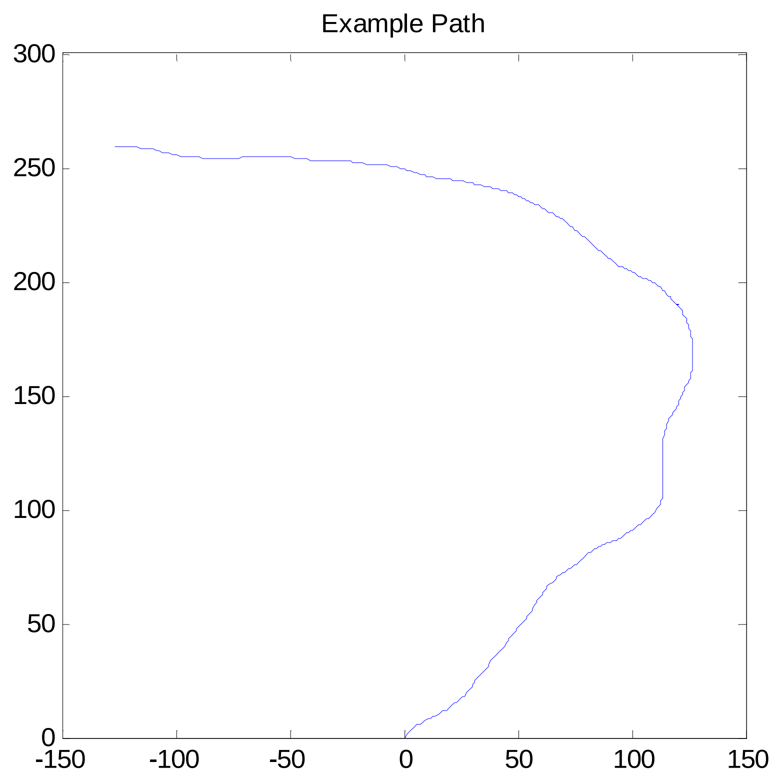
where K_t is known as the Kalman gain.

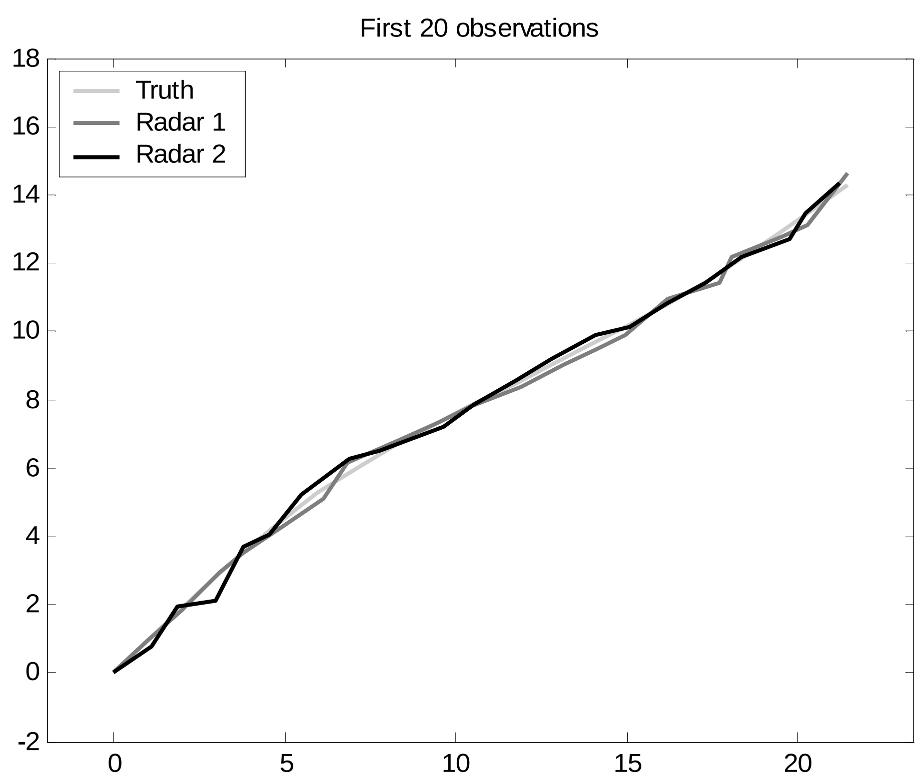
Note that the Kalman filter calculations here reduce to Normal conditional distribution calculations. For example

$$K_t = \mathrm{Cov}(X_t, Z_t)\left(\mathrm{Var}\left(Z_t\right)\right)^{-1},$$

exactly what you need to calculate for a multivariate regression of Z_t on X_t.

P_t reduces to a standard conditional variance calculation.


Example Path
300
250
200
150
100
50
0
-150
-100
-50
0
50
100
150


First 20 observations
Truth
Radar 1
Radar 2
18
16
14
12
10
8
6
4
2
0
-2
0
5
10
15
20

The above data was generated under the model described above with

$$X_0 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$$

$$\Lambda_t = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix}; \ \Lambda_t^{0.5} = \begin{bmatrix} 0.0548 & 0 \\ 0 & 0.0548 \end{bmatrix}$$

and

$$\Sigma_t = \begin{bmatrix} 0.03 & 0 & 0 & 0 \\ 0 & 0.03 & 0 & 0 \\ 0 & 0 & 0.04 & 0.008 \\ 0 & 0 & 0.008 & 0.004 \end{bmatrix}$$

However if the movement or measurement models are non linear or contain non-normal random components, and the Kalman filter or its modifications, such as the Extended Kalman filter (EKF) can give poor answers.

The EKF linearizes the system through Taylor series approximations and then runs the standard Kalman filter on this linear system.

An example with a nonlinear component is given with the movement model

$$
\begin{aligned}
\log s_t &= \log s_{t-1} + \delta_{s,t} \\
\theta_t &= \theta_{t-1} + \delta_{\theta,t} \\
x_t &= x_{t-1} + \frac{s_t \cos\theta_t + s_{t-1}\cos\theta_{t-1}}{2} \\
y_t &= y_{t-1} + \frac{s_t \sin\theta_t + s_{t-1}\sin\theta_{t-1}}{2}
\end{aligned}
$$

In this setting, simulating realizations of X_t will give a better approximation to $\mu_t = E\left[X_t \middle| Z_{1:t}\right]$ and $P_t = \mathrm{Var}\left(X_t \middle| Z_{1:t}\right)$, (or any other functional of X_t).

In this case the distribution of X_t can be extremely difficult to deal with directly, but is fairly easy to deal with conditional of the earlier parts of the path (drawing X_t given X_{t-1} and Z_t is tractable)

For both examples (linkage analysis and target tracking), sequential importance sampling is a useful technique for sampling from the desired posterior distributions.

Let $X = \{X_1, X_2, \ldots, X_k\}$ be some decomposition of the random variable you wish to sample from and $Y = \{Y_1, Y_2, \ldots, Y_k\}$ be the corresponding decomposition of the data you wish to condition on.

Want to sample from

$$p(X|Y) = \frac{p(X)\,p(Y|X)}{p(Y)}$$

which is assumed to be difficult to do.

Want to find a distribution $q(X|Y)$ that is easy to sample from and use importance sampling.

SIS

1) Sample $X_1 \sim q_1(X_1|Y_1)$ and calculate

$$w_1(X_1) = \frac{p(X_1|Y_1)}{q_1(X_1|Y_1)}$$

2) Then for $j = 2, \ldots, k$

Sample $X_j \sim q_j(X_j|Y_{1:j}, X_{1:j-1})$ and calculate

$$w_j\left(X_{1:j}\right) = w_{j-1}\left(X_{1:j-1}\right)$$
$$\times \frac{p\left(X_{1:j} \middle| Y_{1:j}\right)}{q_j\left(X_j \middle| Y_{1:j}, X_{1:j-1}\right) p\left(X_{1:j-1} \middle| Y_{1:j-1}\right)}$$

The factor

$$\frac{p\left(X_{1:j} \middle| Y_{1:j}\right)}{q_j\left(X_j \middle| Y_{1:j}, X_{1:j-1}\right) p\left(X_{1:j-1} \middle| Y_{1:j-1}\right)}$$

is often easy to calculate.

The resulting sample $X = X_{1:k}$ is a weighted sample from $p\left(X \middle| Y\right)$ with unnormalized importance sampling weight

$$w_k\left(X_{1:k}\right) = \frac{p\left(X_{1:k} \middle| Y_{1:k}\right)}{q\left(X_{1:k} \middle| Y_{1:k}\right)}$$

where

$$q\left(X_{1:k} \middle| Y_{1:k}\right) = q_1\left(X_1 \middle| Y_1\right) \prod_{j=2}^{k} q_j\left(X_j \middle| Y_{1:j}, X_{1:j-1}\right)$$

The components of the proposal need to be chosen so that that they are easy to sample from.

Two popular choices are

$$q_j^*\left(X_j \middle| Y_{1:j}, X_{1:j-1}\right) = p\left(X_j \middle| Y_{1:j}, X_{1:j-1}\right)$$

and

$$q_j^{'}\left(X_j \middle| Y_{1:j}, X_{1:j-1}\right) = p\left(X_j \middle| X_{1:j-1}\right)$$

The first choice is optimal in that is minimizes the variance of the importance sampling weights (which will increase the ESS).

The second choice is often easy, such as with the target tracking example. However by ignoring the data, it can significantly increase the importance sampling weight variance.

Optimal proposal properties

For the optimal proposal

$$w\left(X_{1:k}\right) = p\left(Y_1\right)\prod_{j=2}^{k} p\left(Y_j \middle| Y_{1:j-1}, X_{1:j-1}\right)$$

which implies (Kong et al, 1994, Irwin et al, 1994)

$$q^*\left(X_{1:k} \middle| Y_{1:k}\right) = \frac{p\left(Y_{1:k}\right) p\left(X_{1:k} \middle| Y_{1:k}\right)}{w\left(X_{1:k}\right)}$$

or

$$w\left(X_{1:k}\right) = \frac{p\left(Y_{1:k}\right) p\left(X_{1:k} \middle| Y_{1:k}\right)}{q^*\left(X_{1:k} \middle| Y_{1:k}\right)}$$

so

$$\begin{aligned}
E_q\left[w\left(X_{1:k}\right)\right] &= \int w\left(X\right) q^*\left(X_{1:k} \middle| Y_{1:k}\right) dX_{1:k} \\
&= \int w\left(X\right) \frac{p\left(Y_{1:k}\right) p\left(X_{1:k} \middle| Y_{1:k}\right)}{w\left(X_{1:k}\right)} dX_{1:k} \\
&= p\left(Y_{1:k}\right) \int p\left(X_{1:k} \middle| Y_{1:k}\right) dX_{1:k} \\
&= p\left(Y_{1:k}\right)
\end{aligned}$$

Thus the likelihood of the data can be estimated with the average of the unnormalized importance sampling weights.

Implementing SIS for the target tracking example.

Since the movement is described by a Markov chain and the observations are assumed to be independent

$$
\begin{aligned}
q_j^*\left(X_j \middle| Y_{1:j}, X_{1:j-1}\right) &= p\left(X_j \middle| Y_{1:j}, X_{1:j-1}\right) \\
&= p\left(X_j \middle| Y_j, X_{j-1}\right) \\
&\propto p\left(X_j \middle| X_{j-1}\right) p\left(Y_j \middle| X_j\right)
\end{aligned}
$$

So the optimal proposal is tractable here.

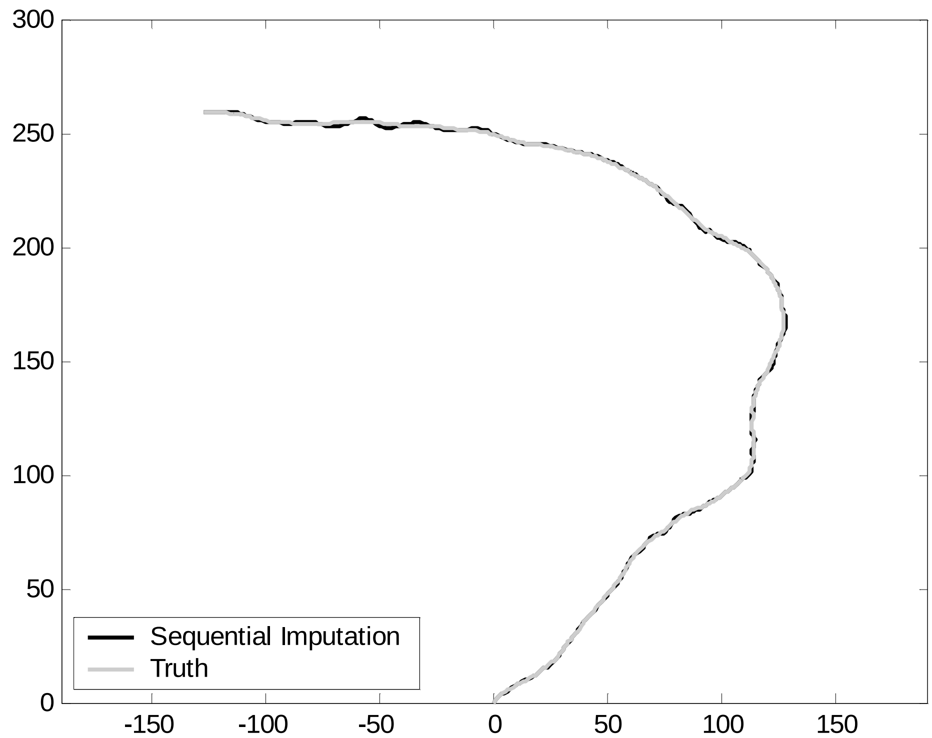
In fact, $X_j \middle| X_{j-1}, Y_j \sim N\left(\theta_j, \Gamma_j\right)$ where

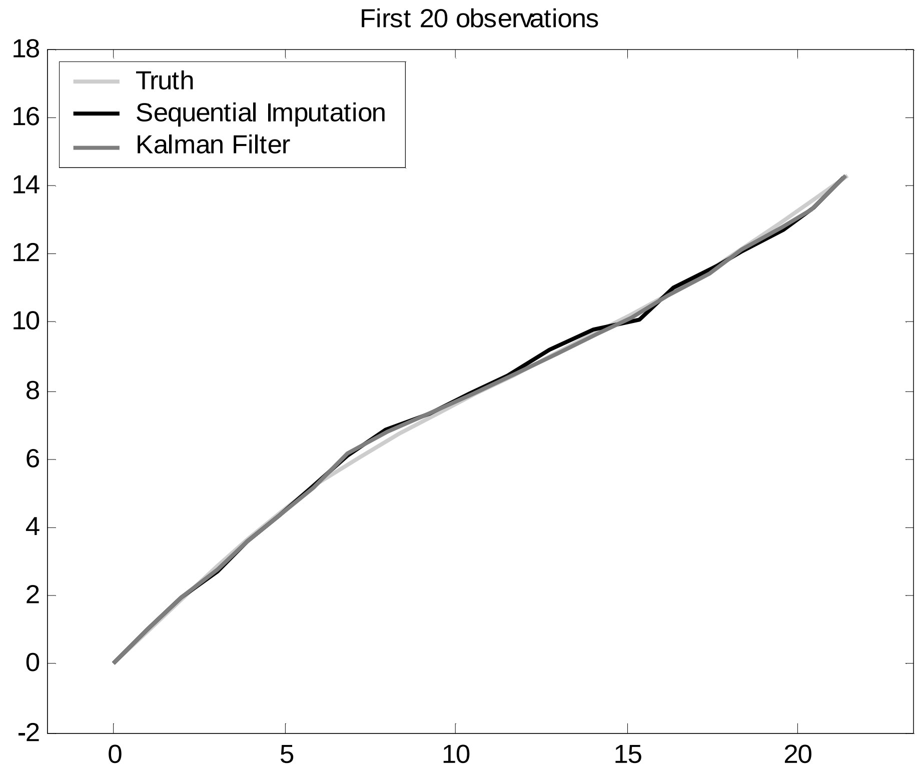
$$
\begin{aligned}
\theta_j &= GX_{j-1} \\
&\quad + H\Lambda_j H^T F^T \left(\Sigma_j + FH\Lambda_j H^T F^T\right)^{-1} \left(Y_j - FGX_{j-1}\right) \\
\Gamma_j &= H\Lambda_j H^T \\
&\quad - H\Lambda_j H^T F^T \left(\Sigma_j + FH\Lambda_j H^T F^T\right)^{-1} FH\Lambda_j H^T
\end{aligned}
$$

In addition, the multiplier for the weight is

$$p\left(Y_j \middle| Y_{1:j-1}, X_{1:j-1}\right) = p\left(Y_j \middle| X_{j-1}\right)$$

which is the density of a $N\left(FGX_{j-1}, \Sigma_j + FH\Lambda_j H^T F^T\right)$ random variable.




First 20 observations
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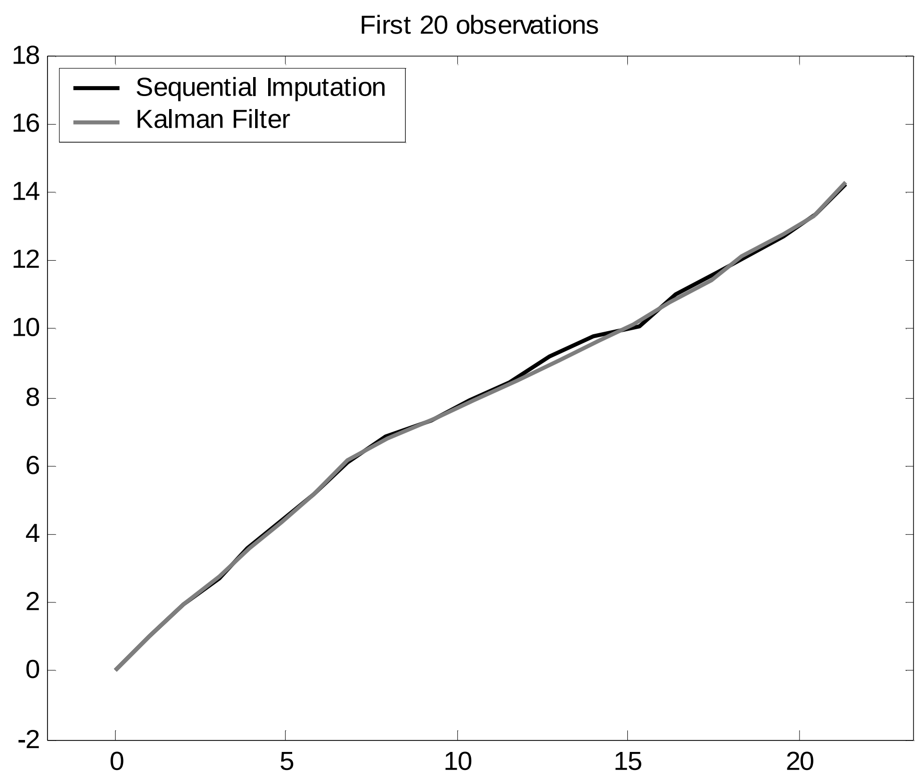
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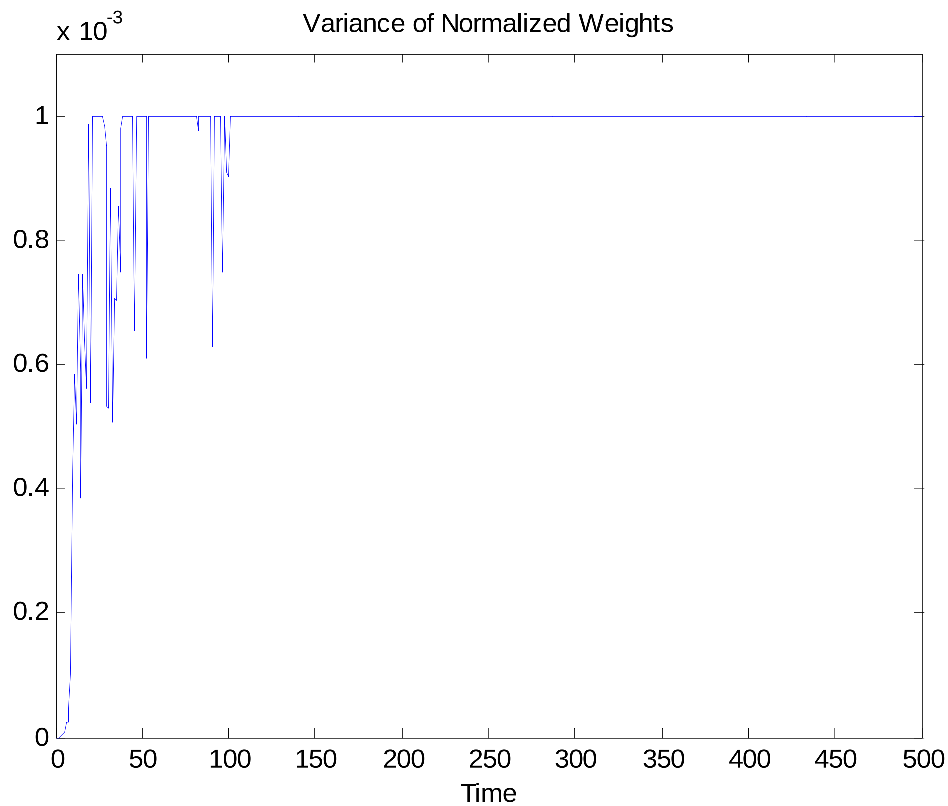
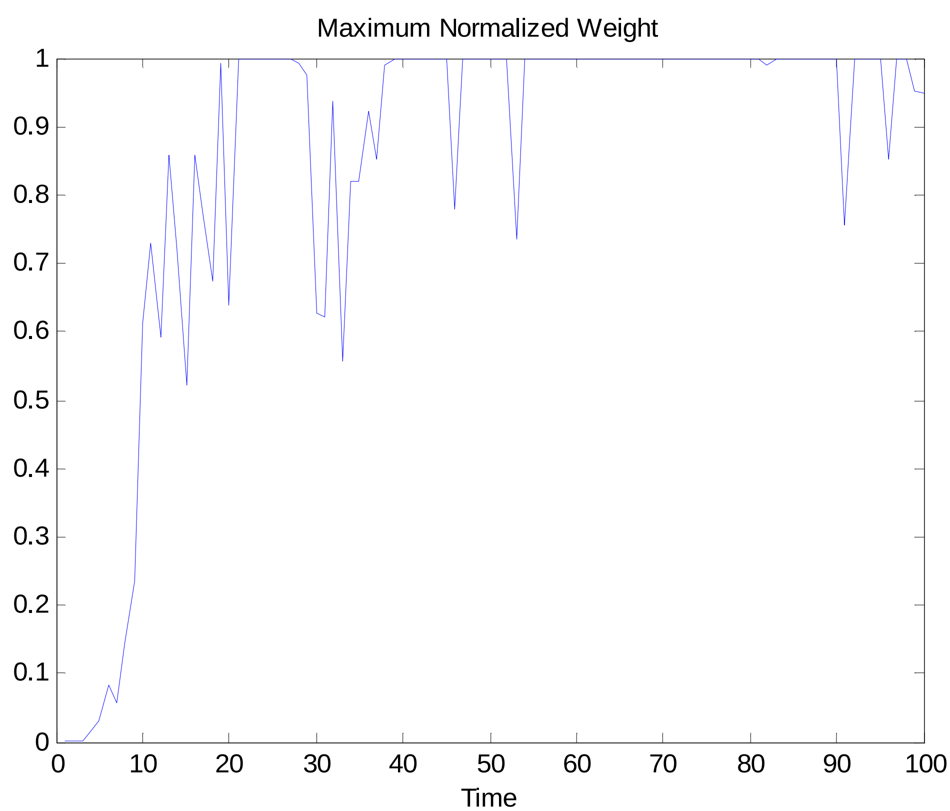

Variance of Normalized Weights
x 10^-3
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
250
300
350
400
450
500
Time


Maximum Normalized Weight
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
80
90
100
Time

One potential problem with SIS is that the variance of the importance sampling weights increases over time, which implies that ESS decreases as the sampler proceeds.

Thus the estimates of the mean are less precise, the further into the sampler we go.

Solution: Resampling.